PHYSICAL REVIEW E VOLUME 49, NUMBER 2 FEBRUARY 1994

Growth instabilities in mechanical breakdown

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(Received 1 June 1993)

A linear stability analysis of a circular crack growing in an elastic medium in two dimensions is presented. Two boundary conditions at the outer boundary are considered, namely, a constant strain and a constant pressure. Size effects are included by assuming a finite distance between the inner and the outer boundaries. If the outer boundary is placed at infinity, the result for the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack is twice that obtained for growth in fields governed by the Laplace equation (diffusion or electrostatic fields) no matter which of the two boundary conditions is imposed. This result is in line with the smaller fractal dimensions obtained in the case of mechanical breakdown.

PACS number(s): 68.70.+w, 05.40.+j, 61.50.Cj

Linear stability analysis has been widely used to illustrate the possibility of instabilities in a large variety of growth problems [1–4]. In particular, these instabilities are responsible for the first stages of the formation of the complex structures found in diffusion-limited aggregation (DLA) [5] and dielectric breakdown (DB) [6] models. Recently the more complex growth problem of mechanical breakdown [7–9] has been also considered from this point of view [10, 11]. Unfortunately in these studies the smooth growing surface was found to be unstable only under rather artificial boundary conditions. For instance, Ref. [10] considered the case of a flat crack assuming the existence of a tangential force at the crack surface. On the other hand, in Refs. [11, 12] it was claimed that a circular crack was only unstable if the crack had a pressure inside, a rather trivial result. It was also stated that no instabilities should be expected in stretched membranes, just the case for which fractal patterns have been obtained [7–9]. It should be pointed out that although in those works a finite threshold was included [11, 12], their conclusions do not change if this threshold vanishes.

In the present work we investigate this question for two boundary conditions that, as shown in Refs. [7–9], produce fractal patterns, namely, a stretched membrane and an external pressure at the outer boundary. We only consider the case of two dimensions (2D). The assumption of a flat crack "front," stretching between the two edges of the system, requires unphysical boundary conditions. In particular, a situation with constant internal pressure cannot be achieved using any sensible boundary condition far from the front. An important ingredient that was overlooked in Refs. [11, 12] is included, namely, a vector normal to the perturbed surface has two nonzero components. This is essential in the case of elasticity due to the tensorial character of the fields [13]. In fact, this is the key point in obtaining the correct answer to the problem. We also include size effects by considering that the crack surface and the outer boundary are not separated by an infinite distance, as has been already done for Laplacian fields [12, 14, 15]. If the outer boundary is placed at infinity, the results indicate that the circular crack is unstable to perturbations of wave number $m > 1$, as in the Laplacian case [1, 5], although in the present case the ratio between the instantaneous rates of growth of the perturbation and that of the circular crack is a factor of 2 larger.

Let us consider a circular crack of radius $r = R_1$ growing in an isotropic elastic medium characterized by the two Lamé coefficients λ and μ [13], at a rate determined by the boundary conditions at an outer boundary R_2 to be defined below. What we shall investigate is the stability of this circular crack upon small perturbations of wave number m (m being a positive integer), such as $r_p = R_1 + \epsilon e^{im\theta}$, where $\epsilon \ll R_1$. Polar coordinates and a polar reference frame will be used hereafter. Once the circular front is perturbed, the most general solution of the Lamé equations [13] can be written as

$$u_r(r,\theta) = v_r(r) + \epsilon U_r(r) e^{im\theta}, \tag{1a}$$

$$u_\theta(r,\theta) = \epsilon U_\theta(r) e^{im\theta}, \tag{1b}$$

where $\mathbf{u}(r,\theta)$ and $v_r(r)$ are the displacement fields in the perturbed and unperturbed cases, respectively. The functions $U_r(r)$ and $U_\theta(r)$ are given by the following expressions:

$$U_r(r) = ar^{1-m} + br^{-1-m} + cr^{m+1} + dr^{m-1}, \tag{2a}$$

$$U_\theta(r) = a\gamma_a r^{1-m} - br^{-1-m} + c\gamma_c r^{m+1} + dr^{m-1}. \tag{2b}$$

The constants γ_a and γ_c in Eqs. (2) are

$$\gamma_a = \frac{m - 4(1-\nu)}{4\nu - 2 - m}, \qquad \gamma_c = \frac{m + 4(1-\nu)}{4\nu - 2 + m}, \tag{3}$$

where $\nu = \lambda/[2(\lambda + \mu)]$ is Poisson's ratio [13]. In writing

Eqs. (2) we have assumed that R_2, the radius of the outer boundary is not infinitely larger than R_1. In this case the positive powers are not unphysical.

Before imposing the boundary conditions at the crack surface we have to write the unit vectors normal ($\mathbf{n}$) and tangential ($\mathbf{t}$) to the crack surface, which, linearizing in ϵ, take the form

$$\mathbf{n} = \left(1, -\frac{i\epsilon m}{R_1} e^{im\theta}\right), \quad \mathbf{t} = \left(\frac{i\epsilon m}{R_1} e^{im\theta}, 1\right). \tag{4}$$

The fact that the normal to the surface has a θ component proportional to ϵ was not taken into account in Refs. [11, 12]. It should be noted that in the case of Laplacian fields this effect gives a second order correction that can be neglected. In the present case, and due to the tensorial nature of the field, this correction is of first order and has to be included. The boundary condition at the crack surface accounts for the fact that no stresses propagate normal to this surface [13]. Both components of the force normal to the surface ($\mathbf{N}$) have to be zero,

$$N_r = \sigma_{rr} n_r + \sigma_{r\theta} n_\theta \approx \sigma_{rr}(r_p) = 0, \tag{5a}$$

$$N_\theta = \sigma_{\theta r} n_r + \sigma_{\theta\theta} n_\theta = \sigma_{\theta r}(R_1) - \frac{i\epsilon m}{R_1} e^{im\theta} \sigma_{\theta\theta}(R_1) = 0, \tag{5b}$$

where σ_{rr}, $\sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are the components of the stress tensor. As regards the outer boundary ($r = R_2$), two boundary conditions are considered in this work, namely, a constant strain (u_0) and a constant pressure (p). In the first case the resulting equations are

$$v_r(R_2) = u_0, \tag{6a}$$

$$U_r(R_2) = U_\theta(R_2) = 0. \tag{6b}$$

Equations (5a) and (6a) give the displacements in the unperturbed case (the index *cs* denotes constant strain)

$$v_r(r) = \beta_{cs}\left[(1-2\nu)r + \frac{R_1^2}{r}\right],$$

$$\beta_{cs} = \frac{u_0}{(1-2\nu)R_2 + \frac{R_1^2}{R_2}}. \tag{7}$$

On the other hand Eq. (6b) combined with Eqs. (5) give the following set of equations for the constants in Eqs. (2):

$$\frac{m(1-m)}{4\nu-2-m} a + (m+1)R_1^{-2} b + \frac{m(m+1)}{4\nu-2+m} R_1^{2m} c + (m-1)R_1^{2m-2} d = \frac{2\beta_{cs} m}{R_1^{1-m}}, \tag{8a}$$

$$\frac{m(m+1)-2}{4\nu-2-m} a - (m+1)R_1^{-2} b + \frac{m(m-1)-2}{4\nu-2+m} R_1^{2m} c + (m-1)R_1^{2m-2} d = -\frac{2\beta_{cs}}{R_1^{1-m}}, \tag{8b}$$

$$a + R_2^{-2} b + R_2^{2m} c + R_2^{2m-2} d = 0, \tag{8c}$$

$$\gamma_a a - R_2^{-2} b + \gamma_c R_2^{2m} c + R_2^{2m-2} d = 0. \tag{8d}$$

In the case of a constant pressure p at the outer boundary Eqs. (6) are replaced by

$$\sigma_{rr}(R_2) = p, \qquad \sigma_{r\theta}(R_2) = 0. \tag{9}$$

As a result the constant β_{cs} in Eq. (7) is replaced by

$$\beta_{cp} = \frac{pR_2^2}{(1-2\nu)(R_2^2 - R_1^2)} \tag{10}$$

where the index *cp* denotes constant pressure, and the set of equations that give the four constants in Eqs. (2) has to be modified as follows. In the first two we only need to replace β_{cs} by β_{cp}, whereas the third and the fourth are obtained by replacing in Eqs. (8a) and (8b) R_1 by R_2 and the right-hand side by zero.

As in Refs. [7–9] we assume that the growth rate is proportional to the modulus of the tangential tension ($\mathbf{T}$). This is easily calculated from the stress tensor calculated at the crack surface ($r = r_p$) and the tangential vector given in Eq. (4). The result is

$$\mathbf{T} = \sigma_{\theta\theta}(r_p)\left(\frac{i\epsilon m}{R_1} e^{im\theta}, 1\right). \tag{11}$$

Thus the instantaneous growth rate is given by

$$\dot{R}_1 + \dot{\epsilon} e^{im\theta} = C\sigma_{\theta\theta}(r_p), \tag{12}$$

where C is a constant. Then the result for the ratio between the instantaneous rates of growth of the perturbation ($\dot{\epsilon}$) and that of the disk ($\dot{R}_1$) in the case of constant strain is

$$\alpha_m = 2(m-1) - \frac{2R_1^{m+1}}{\beta_{cs}}\left[\frac{m(m+1)}{4\nu-2+m} c + (m-1)R_1^{-2} d\right]. \tag{13}$$

In the case of constant pressure β_{cs} should be replaced by β_{cp}, and the constants c and d by those corresponding to this boundary condition. It is interesting to note that if the outer boundary is placed at infinity, the result for α_m is

$$\alpha_m = 2(m-1) \tag{14}$$

for the two boundary conditions here considered. We note that α_m is twice that found in the case of growth in fields governed by the Laplace equation (DLA and DB) [1, 5]. This result is in accordance with an analysis of the field singularities along the lines of Ref. [16]. In fact, the singularities that appear at wedges in an elastic medium [10, 17] are stronger than those found in Laplacian fields [16]. As a consequence the predicted fractal dimensions for the elastic case [8] are smaller than those obtained in Ref. [16] for Laplacian fields, in agreement with numerical results.

We have studied the case of a finite R_2 by numerically solving Eqs. (8) for the constants a–d, and substituting the results for c and d in Eq. (13). The results for α_m are shown in Fig. 1. The following features are worthy of comment: (i) the results for constant strain are always below the asymptotic value of Eq. (14), whereas the opposite holds for constant pressure; (ii) the asymptotic value is reached faster as m increases; and (iii) as R_2 tends to R_1, α_m increases up to ∞, in the case of a constant pressure, whereas for constant strain it decreases to $-\frac{4}{3}$; in both cases these values are independent of m. It is interesting to compare these results with those obtained for Laplacian fields. In the latter case α_m is given by

$$\alpha_m = m\frac{(R_2/R_1)^{2m} \pm 1}{(R_2/R_1)^{2m} \mp 1} - 1, \tag{15}$$

where $\pm$ signs correspond to fixing either the potential or its derivative at the outer boundary (Dirichlet or von Neumann boundary conditions). We note that this equation shows a behavior similar to that found for α_m in the case of elasticity. For instance, for $R_2/R_1 = 1+\epsilon$, where $\epsilon \ll 1$, α_m tends to either $1/\epsilon - 1$ (for all m) or $m^2\epsilon - 1$, for either constant potential or constant field. We also note that the results for α_m obtained in the case of elasticity (see Fig. 1) are always larger than the values given by Eq. (15) except for R_2 very close to R_1 and fixed strain at the outer boundary (as an example we note that for m=2 and m=4 this occurs for $R_2 < 1.05R_1$ and $R_2 < 1.02R_1$, respectively). Thus, only in a range of R_2 of minor interest, the instabilities in Laplacian fields can be stronger than in the case of elasticity.

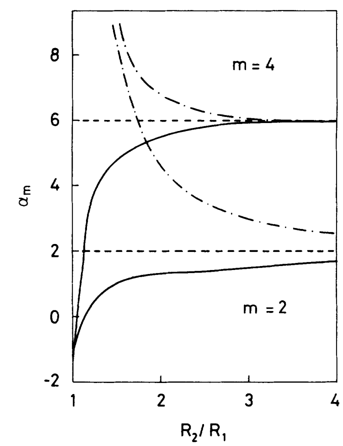

FIG. 1. Results for the ratio between the instantaneous growth rates of the perturbation and that of the circular crack ($\alpha_m, m = 2,4$) as a function of R_2/R_1, for $\nu = 0.25$ ($\lambda = \mu$) and the two boundary conditions at the outer boundary ($r = R_2$) considered in this work, namely, constant strain (continuous lines) and constant pressure (chain lines). At $R_2 = R_1$, the values of α_m for constant strain and pressure are $-\frac{4}{3}$ and ∞, respectively. The horizontal chain lines indicate the values of α_m for the outer boundary at infinity.

In conclusion, we have presented a study of growth instabilities in mechanical breakdown and obtained results that are similar to those already found for growth in Laplacian fields, and in variance with previous studies of the problem [11]. The most important conclusion is that growth in an elastic medium results in being more prone to instabilities of the Mullins-Sekerka type than growth in Laplacian fields. This is in line with the smaller fractal dimensions found in numerical simulations of mechanical breakdown [7–9] as compared to those obtained in the cases of DLA [5] and dielectric breakdown [6].

We wish to acknowledge financial support from the Spanish CICYT Grant No. MAT91-0905-C02.

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